9/05/2019

(Απειροστικός 2
Σαρρίζαλι)

$$\int \frac{1}{\cos\theta}\,d\theta = \ln\left|\frac{1}{\cos\theta} + \tan\theta\right| + c$$

ΠΑΡΑΔΕΙΓΜΑ 4: $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$\alpha^2 - x^2 \rightarrow x = \alpha\sin\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$ $\quad x+1 = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$

$\alpha^2 + x^2 \rightarrow x = \alpha\tan\theta$, -ll-

$x^2 - \alpha^2 \rightarrow x = \frac{\alpha}{\cos\theta}$, $0 < \theta < \frac{\pi}{2}$

$$\int \frac{2\sin\theta - 1}{\sqrt{4(1-\sin^2\theta)}}\, 2\cos\theta\,d\theta$$

$$\overset{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}{=} \int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}}\,\cancel{2\cos\theta}\,d\theta = -2\cos\theta - \theta + c$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x+1}{2}\right)^2}$$

$$2\sin\theta = x+1 \Rightarrow \theta = \operatorname{Arcsin}\frac{x+1}{2}$$

$$\Rightarrow \int \frac{x}{\sqrt{3-2x-x}}\,dx = -2\sqrt{1-\left(\frac{x+1}{2}\right)^2} - \operatorname{Arcsin}\frac{x+1}{2} + c$$

ΡΗΤΕΣ ΣΥΝΑΡΤΗΣΕΙΣ ΤΩΝ cosx, sinx, c

Π.χ. $\int \frac{\cos x + \sin x}{\cos^2 x}\,dx$, $\int \frac{2\cos^2 x + 1 - \sin x}{\sin^3 x + 1}\,dx$

ΣΥΧΝΗ ΑΝΤΙΚΑΤΑΣΤΑΣΗ: $u = \tan\frac{x}{2}$ $\quad du = \frac{1}{2}\cdot\frac{1}{\cos^2(x/2)}\,dx = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)dx$

$= \frac{1+u^2}{2}\,dx \Rightarrow dx = \frac{2}{1+u^2}\,du$

$$\boxed{\begin{array}{l}\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \\ \sin 2\alpha = 2\sin\alpha\cos\alpha\end{array}}$$

$\cos x = \cos\left(2\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \dfrac{\cos^2 x/2 - \sin^2 x/2}{\cos^2 x/2 + \sin^2 x/2}$

$= \dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2} = \dfrac{1-u^2}{1+u^2}$ $\quad \boxed{\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = 2u}$

ΠΑΡΑΔΕΙΓΜΑ: $\displaystyle\int \frac{1-\sin x}{1+\cos x}\,dx \xrightarrow{u=\tan x/2} \int \frac{1-2u(u^2+1)}{1+\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du$

$\boxed{\tan^2\frac{x}{2} + 1 = \cos^2\frac{x}{2}}$

$= \displaystyle\int \frac{4(1-2u(u^2+1))}{(1+u^2)^2}\,du$

ΟΛΟΚΛΗΡΩΣΗ ΡΗΤΩΝ ΣΥΝΑΡΤΗΣΕΩΝ

$\displaystyle\int \frac{P(x)}{q(x)}\,dx$, $P(x), q(x)$ πολυώνυμα

1ο Βήμα: Ευκλείδια διαίρεση
$\exists\, \pi(x), u(x)$ με ~~...~~ $\deg(u(x)) < \deg(q(x))$ τω
$P(x) = \pi(x)\,q(x) + u(x)$

Άρα $\displaystyle\int \frac{P(x)}{q(x)}\,dx = \int \left[\pi(x) + \frac{u(x)}{q(x)}\right] dx$

Αναγώμαστε στον υπολογισμό του $\displaystyle\int \frac{u(x)}{q(x)}\,dx$, όπου

$\deg(u(x)) < \deg(q(x))$

$$\int \frac{Bx+\Gamma}{(x^2+bx+\gamma)^k}\,dx = \frac{B}{2}\int \frac{2x+b}{(x^2+bx+\gamma)^k}\,dx + \left(\Gamma - \frac{Bb}{2}\right)\int \frac{1}{(x^2+bx+\gamma)^k}\,d$$

$$y = x^2+bx+\gamma$$

$$dy = (2x+bx)\,dx$$

$$= 2\left(x+\frac{b}{2}\right)dx$$

$$I_1 = \int \frac{1}{y^k}\,dy = -(k+1)\,y^{-(k+1)} + c$$

I_2 $\underline{\text{Συμπλήρωση}}$ τετραγώνου $\displaystyle\int \frac{1}{\left[\left(x+\frac{b}{2}\right)^2 + \frac{4\gamma-b^2}{4}\right]^k}\,dx$

$$= \quad x+\frac{b}{2} = \sqrt{\frac{4\gamma-b^2}{4}}\;y$$

$$= \int \frac{1}{\left(\frac{4\gamma-b^2}{4}y^2 + \frac{4\gamma-b^2}{4}\right)^k}\cdot\sqrt{\frac{4\gamma-b^2}{4}}\,dy$$

$$= \frac{1}{\left(\frac{4\gamma-b^2}{4}\right)^{k-1/2}}\int \underbrace{\frac{1}{(1+y^2)^k}dy}_{I_k}$$

$$\boxed{I_{k+1} = \frac{1}{2k}\,\frac{y}{(y^2+1)^k} + \frac{2k-1}{2k}\,I_k}$$

ΠΑΡΑΔΕΙΓΜΑ $A(x) = \int \frac{3x+4}{(x^2+2x+5)^2} dx$

$$= \int \frac{3x+3}{(x^2+2x+5)^2} dx + \underbrace{\int \frac{1}{(x^2+2x+5)^2} dx}_{I_0}$$

$$= \frac{3}{2} \int \frac{1}{y^2} dy = \frac{3}{2} \frac{-y^{-3}}{3} + c = -\frac{y^{-3}}{2} + c$$

$$I_0 = \int \frac{1}{[(x+1)^2+4]^2} dx \xrightarrow[dx=2dy]{x+1=2y} \int \frac{2}{16(y^2+1)} dy = \frac{1}{8} I_2$$

$$\boxed{\begin{aligned} I_2 &= \frac{1}{2} \frac{y}{(y^2+1)} + \frac{1}{2} I_1 \\ &= \frac{1}{2} \frac{y}{y^2+1} + \frac{1}{2} \int \frac{1}{y^2+1} dy \\ &= 1/2 \frac{y}{y^2+1} + \frac{1}{2} \mathrm{Arctan}\, y \end{aligned}}$$

$$= \frac{1}{16} \frac{(x+1)/2}{\left(\frac{x+1}{2}\right)^2 + 1} + \frac{1}{2} \mathrm{Arctan} \frac{x+1}{2} + c$$

2ο Βήμα $\int \frac{P(x)}{q(x)} dx$, $\deg(P(x)) < \deg(q(x))$

ΘΕΩΡΗΜΑ Κάθε πολυώνυμο γράφεται ως γινόμενο πολυωνύμων βαθμού ένα και πολυωνύμων βαθμού δύο χωρίς πραγματικές ρίζες.

$q(x) = C(x-\alpha_1)^{r_1} \ldots (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \ldots$
$(x^2+\beta_\ell x+\gamma_\ell)^{s_\ell}$

$\beta_i^2 - 4\gamma_i < 0 \quad i=1,\ldots,\ell$

$\alpha_1 \ldots \alpha_k$: Οι πραγματικές ρίζες του $q(x)$

Γράφω το $\dfrac{P(x)}{q(x)} = \left[\dfrac{A_{11}}{x-\alpha_1} + \dfrac{A_{12}}{(x-\alpha_1)^2} + \ldots + \dfrac{A_{1r_1}}{(x-\alpha_1)^{r_1}}\right] +$

$$+\left[\frac{A_{k1}}{x-\alpha_k} + \frac{A_{k2}}{(x-\alpha_k)^2} + \ldots + \frac{A_{kr_k}}{(x-\alpha_k)^{r_k}}\right]$$

$$+\left[\frac{B_{11}x+\Gamma_{11}}{x^2+\beta_1 x+\gamma_1} + \frac{B_{12}x+\Gamma_{12}}{(x^2+\beta_1 x+\gamma_1)} + \ldots + \frac{B_{1s_1}x+\Gamma_{1s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right] + \ldots$$

$$+\left[\frac{B_{\ell 1}x+\Gamma_{\ell 1}}{x^2+\beta_\ell x+\gamma_\ell} + \frac{B_{\ell s_\ell}x+\Gamma_{\ell s_\ell}}{(x^2+\beta_\ell x+\gamma_\ell)^{s_\ell}}\right]$$

Αναγόμαστε στον υπολογισμό ολοκληρωμάτων της μορφής

$$\int \frac{1}{(x-\alpha)^k}\,dx, \quad \int \frac{Bx+\Gamma}{(x^2+\beta x+\gamma)^k}\,dx$$

ΠΑΡΑΔΕΙΓΜΑΤΑ 1) $\displaystyle\int \frac{3x^2+6}{x^3+x^2-2x}\,dx = \int \frac{3x^2+6}{x(x-1)(x+2)}\,dx$

$$\frac{3x^2+6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{\Gamma}{x+2}$$